Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 269 (2004) 623–632
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Optimal damper location for randomly forced cantilever beams

Raymond G. Jacquot

Electrical and Computer Engineering Department, University of Wyoming, Box 3295, University Station, Laramie, WY 82071, USA

Received 4 April 2002; accepted 14 December 2002

Abstract

The work reported here considers the attachment of a viscous damper to a cantilever beam that is driven by temporally white noise with several different spatial distributions. The variables in the problem to be optimized are the location and the value of the damper. For each of the spatial distributions, there is a location and damper value that will minimize the mean square motion averaged over the beam length. These minima are shown to be not a strong function of the spatial distribution of the forcing function with the best location being at 70% of the length from the fixed end. The optimal value of the damper is shown to be 50 times the product of the beam mass and the first radian natural frequency for the beam.

1. Introduction

It is obvious that the incorporation of damping or passive energy dissipation elements into vibratory systems is a means of limiting unwanted vibration. In single degree-of-freedom systems, the analysis of how much damping to add will depend on whether the vibration to be minimized is forced or transient. In multiple degree-of-freedom systems, the picture is not so clear in that the locations at which damping may be added are new variables in the problem in addition to the amounts of damping to be added at each location. If excessive damping is added between a given mass and inertial ground that mass will have small vibration but this will result in large vibration at other masses in the system. A similar effect occurs with dissipative connection between masses of a system. The two issues here are: (1) where to add the damping and (2) how much damping to add?

Much work has been done to predict system response when damping has been added to a conservative system for which the unmodified modes and natural frequencies are known [1–5] but little has been done to find both optimal location, distribution and the value of the damping. The

E-mail address: quot@uwyo.edu (R.G. Jacquot).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00073-7

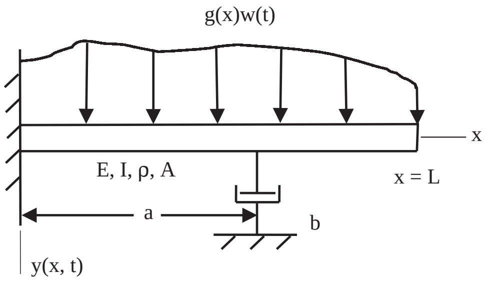

Fig. 1. Cantilever beam with a variable-in-space, random-in-time forcing function with an arbitrarily located viscous damper.

answer to the above-posed question is not easily given and depends on whether the vibration is free or forced and, if forced, whether transient, periodic or random.

Many investigators have considered the random response of damped vibratory systems [6–12]. In a previous paper, the author has shown that a simple measure of overall random response of a damped distributed parameter system is the spatially averaged mean square response [12]. In the same work, it was shown that for a viscous damper attached at the cantilever tip there is a damper value that minimizes the spatially averaged mean square response. In that work, it was also shown that an optimal dynamic vibration absorber applied at the tip could be found, however, the resulting average mean square response was considerably higher than that for the optimal damper to ground.

Of recent interest in the structural engineering community is the damping of vibration in stay cables such as that are incorporated into bridge structures. Pacheco et al. [13] have explored the use of passive viscous damping near the cable anchorage to suppress vibration and a dimensionless design curve is developed. Viscous dampers installed near the anchorage of cables have been shown to be of limited effectiveness and hence the concept of semiactive damping has been explored by Johnson et al. [14].

In this work, the author considers the question posed above for the Bernoulli–Euler cantilever beam with a single added viscous damper as illustrated in Fig. 1. This beam is a relatively simple system for which the mode shapes and natural frequencies are well known [15]. The beam is assumed to be driven by a stationary, zero mean, random in time force with three quite different spatial distributions. These distributions are uniform in space, increasing with location and decreasing with location as illustrated in Fig. 2.

2. Theory

2.1. Transfer functions

In Ref. [10] the infinite series form transfer function between the damper force $P(s)$ and the response at some point x on the beam is given as

$$G_1(x,s) = \frac{Y(x,s)}{P(s)} = \frac{1}{\rho AL}\sum_{i=1}^{\infty}\frac{\phi_i(a)\phi_i(x)}{s^2+\omega_i^2}, \tag{1}$$

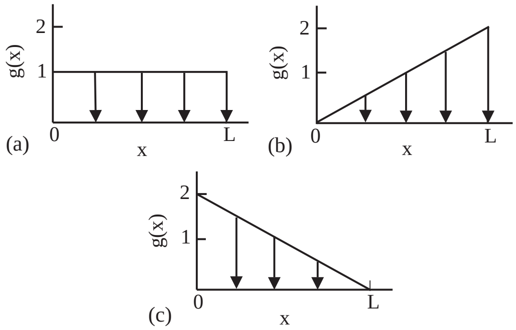

Fig. 2. Three spatial forcing functions considered in the three cases of this work: (a) a uniform spatial load, (b) a linearly increasing load and (c) a linearly decreasing load.

where ρA is the mass per unit length, the $\phi_i(x)$ are the orthogonal beam functions associated with the clamped-free beam given as

$$\phi_i(x) = \cosh \beta_i x - \cos \beta_i x - \alpha_i(\sinh \beta_i x - \sin \beta_i x), \tag{2}$$

and the square of the ith natural frequency is $\omega_i^2 = \beta_i^4 EI/\rho A$. The values of α_i and $\beta_i L$ have been tabulated by Young [16] for beams with various boundary conditions.

The driving force is assumed to be distributed and of the form $g(x)w(t)$ and thus it is appropriate to expand $g(x)$ in a generalized Fourier series in the cantilever eigenfunctions. This nature of this type of forcing function does not lend itself wind or turbulent fluid flow loadings. The transfer function between $W(s)$ and the response at some point x on the beam without the presence of the damper is similarly

$$G_2(x,s) = \frac{Y(x,s)}{W(s)} = \sum_{i=1}^{\infty} \frac{d_i \phi_i(x)}{\rho A(s^2 + \omega_i^2)}, \tag{3}$$

where the d_i depend on the spatial distribution of the applied force $g(x)$ according to

$$d_i = \frac{1}{L}\int_0^L g(x)\phi_i(x)\,\mathrm{d}x. \tag{4}$$

It should be noted that transfer functions (1) and (3) are open-loop transfer functions. In order to investigate the effect of the spatial distribution of the random forcing function, three different spatial distribution functions $g(x)$ will be used here and will be referred to as cases (a), (b) and (c). These three distributions are illustrated in Fig. 2 and each has an area equal to the length of the beam.

Case (a): When the force is uniform $g(x) = 1$ as in Fig. 2(a) the coefficient d_i is given by the integral tables of Felgar [16] to be

$$d_i = \frac{2\alpha_i}{\beta_i L}. \tag{5}$$

Table 1
Expansion coefficients d_i for the three different spatial force distributions

Mode i	Case		
	(a)	(b)	(c)
1	0.7830	1.1377	0.4283
2	0.4339	0.1815	0.6863
3	0.2544	0.0648	0.4440
4	0.1819	0.0331	0.3307
5	0.1415	0.0200	0.2629

Case (b): Similarly, when the force is uniformly increasing $g(x) = 2x/L$ as shown in Fig. 2(b) the coefficient is given by integral (4) to be

$$d_i = \frac{4}{(\beta_i L)^2}. \tag{6}$$

Case (c): Lastly, when the force distribution is as illustrated in Fig. 2(c) $g(x) = 2[1 - (x/L)]$ the coefficients d_i are

$$d_i = \frac{4\alpha_i}{\beta_i L} - \frac{4}{(\beta_i L)^2}. \tag{7}$$

The first five of these coefficients have been calculated for each of the three cases and are given in Table 1.

Inspection of these expansion coefficients indicates that the first mode is the most highly driven in cases (a) and (b) while in case (c) the second mode is the most highly driven.

In Ref. [10] the transfer function between the applied force $W(s)$ and the response at the point of the damper attachment $x = a$ was given to be

$$M(a,s) = \frac{Y(a,s)}{W(s)} = \frac{G_2(a,s)}{1 + G_1(a,s)H(s)}, \tag{8}$$

where $H(s) = bs$ is the displacement driving point impedance of the damper. Also in Ref. [10] the transfer function between $W(s)$ and the response at any point x on the beam is

$$M(x,s) = \frac{Y(x,s)}{W(s)} = G_2(x,s) - G_1(x,s)M(a,s)H(s), \tag{9}$$

where $M(a,s)$ is defined by relation (8).

2.2. Spatially averaged mean square response

The power spectral density of the response at some point x is related to the power spectral density of $w(t)$, $S_w(\omega)$, by the harmonic response transfer function $M(x, j\omega)$ according to

$$S_y(x,\omega) = |M(x,j\omega)|^2 S_w(\omega). \tag{10}$$

If the power spectral density of $y(x,t)$ is known from Eq. (10) then the mean square motion at a location x is proportional to the area under that power spectrum or

$$\sigma_y^2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(x,\mathrm{j}\omega)|^2 S_w(\omega)\,\mathrm{d}\omega. \tag{11}$$

In many cases of vibration control, the object is to limit vibration throughout the structure, or in case of a distributed parameter system, over the spatial domain of the structure. For the beam that domain is $0 \leqslant x \leqslant L$. It was shown in Ref. [12] that one such measure of gross response of a beam is the spatially averaged mean square response or

$$\overline{\sigma_y^2} = \frac{1}{L}\int_0^L \sigma_y^2(x)\,\mathrm{d}x = \frac{1}{2\pi L}\int_0^L \int_{-\infty}^{\infty} |M(x,\mathrm{j}\omega)|^2 S_w(\omega)\,\mathrm{d}\omega\,\mathrm{d}x. \tag{12}$$

It is known from Ref. [12] that for a damper applied at the beam tip, a value of the damper parameter b exists which minimizes the spatially averaged mean square motion. The questions to be answered here are whether a damper attached at some other point will again minimize the average mean square motion and whether there is some location which will yield a global minimum average mean square motion and if so what will be the damper value?

2.3. Frequency scaled transfer functions

In order to present results in a general fashion it makes sense to get the transfer functions in terms of a dimensionless frequency variable. In order to accomplish this scale the s-domain transform variable as

$$s = \omega_1 p. \tag{13}$$

With this definition the transfer functions are, respectively,

$$G_1(x,p) = \frac{1}{\rho A L \omega_1^2}\sum_{i=1}^{\infty}\frac{\phi_i(x)\phi_i(a)}{p^2+\gamma_i^2}, \tag{14}$$

$$G_2(x,p) = \frac{L^4}{EI(\beta_1 L)^4}\sum_{i=1}^{\infty}\frac{d_i\phi_i(x)}{p^2+\gamma_i^2}, \tag{15}$$

$$H(p) = b\omega_1 p, \tag{16}$$

where the dimensionless natural frequency variable is $\gamma_i = \omega_i/\omega_1$. If these relations are substituted into relation (8) the result for $M(a,p)$ is

$$M(a,p) = \frac{\dfrac{L^4}{EI(\beta_1 L)^4}\sum_{i=1}^{\infty}\dfrac{d_i\phi_i(a)}{p^2+\gamma_i^2}}{1 + Cp\sum_{i=1}^{\infty}\dfrac{\phi_i^2(a)}{p^2+\gamma_i^2}}, \tag{17}$$

where the dimensionless damping coefficient is $C = b/\rho AL\omega_1$. If this relation is further substituted into relation (9) the result is

$$M(x,p) = \frac{L^4}{EI(\beta_1 L)^4}\sum_{n=1}^{\infty}\left[d_n - Cp\phi_n(a)\left(\frac{\sum_{i=1}^{\infty}\dfrac{d_i\phi_i(a)}{p^2+\gamma_i^2}}{1 + Cp\sum_{i=1}^{\infty}\dfrac{\phi_i^2(a)}{p^2+\gamma_i^2}}\right)\right]\frac{\phi_n(x)}{p^2+\gamma_n^2}. \tag{18}$$

More important to the stationary random vibration problem is the frequency response function which is given by substituting jω for s in the transfer function. Having scaled the transfer function variable s, it is thus appropriate to deal in scaled frequency substituting jf for p where $f = \omega/\omega_1$. The result is

$$M(x,\mathrm{j}f) = \frac{L^4}{EI(\beta_1 L)^4}\sum_{n=1}^{\infty}\left[d_n - \mathrm{j}fC\phi_n(a)\left(\frac{\sum_{i=1}^{\infty}\dfrac{d_i\phi_i(a)}{\gamma_i^2-f^2}}{1 + \mathrm{j}fC\sum_{i=1}^{\infty}\dfrac{\phi_i^2(a)}{\gamma_i^2-f^2}}\right)\right]\frac{\phi_n(x)}{\gamma_n^2-f^2}. \tag{19}$$

Define the coefficient $\varepsilon_n(a,f)$ as

$$\varepsilon_n(a,f) = \frac{1}{(\beta_1 L)^4}\left[d_n - \mathrm{j}fC\phi_n(a)\left(\frac{\sum_{i=1}^{\infty}\dfrac{d_i\phi_i(a)}{\gamma_i^2-f^2}}{1 + \mathrm{j}fC\sum_{i=1}^{\infty}\dfrac{\phi_i^2(a)}{\gamma_i^2-f^2}}\right)\right]. \tag{20}$$

Then, relation (19) can be written as

$$M(x,\mathrm{j}f) = \frac{L^4}{EI}\sum_{n=1}^{\infty}\frac{\varepsilon_n(a,f)\phi_n(x)}{\gamma_n^2-f^2}. \tag{21}$$

The mean square motion at point x is given by Eq. (11) which in terms of the scaled frequency variable f is

$$\sigma_y^2(x) = \frac{\omega_1}{2\pi}\int_{-\infty}^{\infty}|M(x,\mathrm{j}f)|^2 S_w(f)\,\mathrm{d}f. \tag{22}$$

If $w(t)$ is assumed to be white noise with constant power spectral density function S_w the average mean square motion from Eq. (12) is

$$\frac{\overline{\sigma_y^2}(EI)^2}{\omega_1 L^8 S_w} = \frac{1}{2\pi L}\int_0^L\int_{-\infty}^{\infty}|M(x,\mathrm{j}f)|^2\,\mathrm{d}f\,\mathrm{d}x. \tag{23}$$

Relation (21) may be substituted to yield

$$\frac{\overline{\sigma_y^2}(EI)^2}{\omega_1 L^8 S_w} = \frac{1}{2\pi L}\int_0^L\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{\varepsilon_n(a,f)\phi_n(x)}{\gamma_n^2-f^2}\sum_{k=1}^{\infty}\frac{\varepsilon_k^*(a,f)\phi_k(x)}{\gamma_k^2-f^2}\,\mathrm{d}f\,\mathrm{d}x, \tag{24}$$

where the * denotes the complex conjugate. If the spatial integral is evaluated noting the orthogonality of the modes the result is

$$\frac{\overline{\sigma_y^2}(EI)^2}{\omega_1 L^8 S_w} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{|\varepsilon_n(a,f)|^2}{(\gamma_n^2 - f^2)^2}\,\mathrm{d}f \tag{25}$$

and since the integrand of Eq. (25) is an even function of frequency the integration can be accomplished for only positive frequency or

$$\frac{\overline{\sigma_y^2}(EI)^2}{\omega_1 L^8 S_w} = \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{|\varepsilon_n(a,f)|^2}{(\gamma_n^2 - f^2)^2}\,\mathrm{d}f, \tag{26}$$

where the $\varepsilon_n(a,f)$ are defined by relation (20). This is the function to be minimized with respect to the parameters a, the location of the damper, and C, the dimensionless coefficient of damping for the device.

3. Discussion of results

All the three cases have been explored here by evaluation of expression (26) employing MATLAB. The series involved were truncated at both four and five terms with no appreciable difference in the result because in all three cases most of the response energy is in the first three modes. For case (a) the spatial average mean square motion as a function of the damping parameter C for various values of damper location is illustrated in Fig. 3. Note that for each

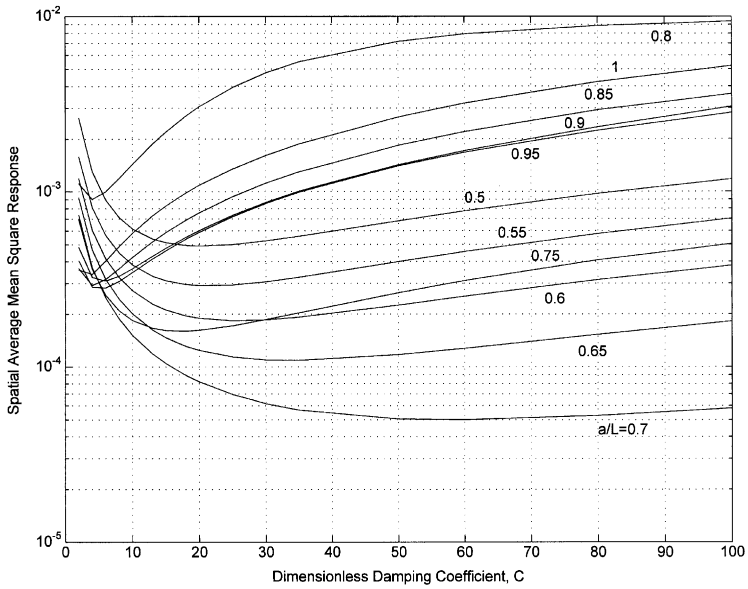

Fig. 3. Spatial average mean square response for a variety of damper locations as a function of damper parameter C for case (a).

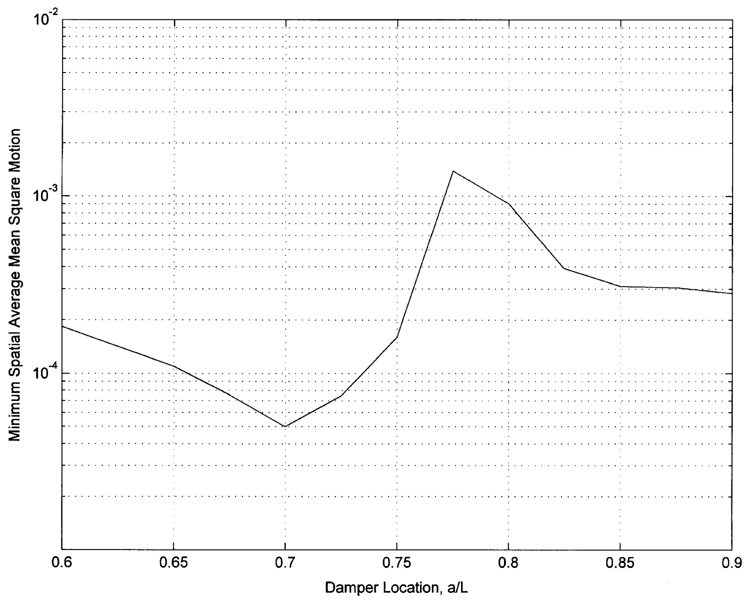

Fig. 4. Minimum spatial average mean square motion as a function of damper location for case (a).

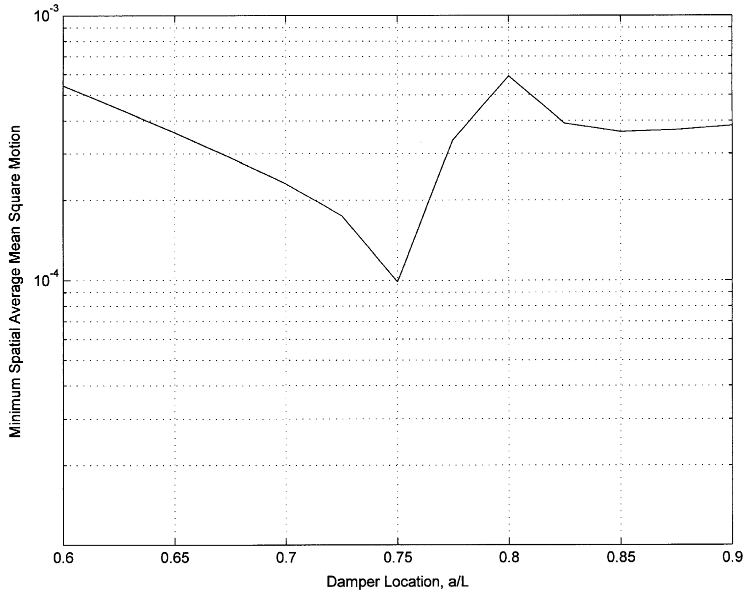

Fig. 5. Minimum spatial average mean square motion as a function of damper location for case (b).

damper location there is a minimum occurring at a different values of dimensionless damper parameter C. For damper locations near the tip the optimal damping parameter is between three and five. As the damper is moved toward the root of the beam, the optimal viscosity increases by an order of magnitude. On reflection it is seen that this dramatic increase in the optimal damping is such to make the damper a nearly rigid support which decreases the mean square motion while

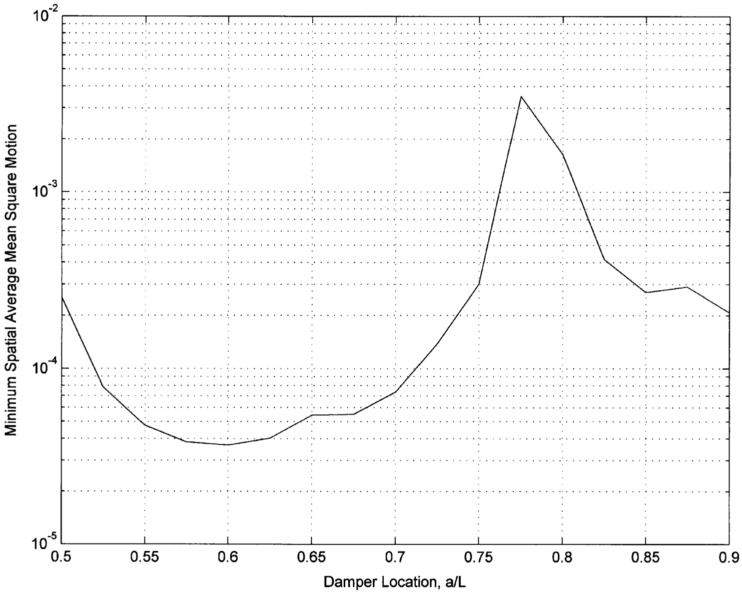

Fig. 6. Minimum spatial average mean square motion as a function of damper location for case (c).

Table 2
Optimal damper locations and parameters for the three cases

Case	(a)	(b)	(c)
$(a/L)_{opt}$	0.7	0.75	0.6
C_{opt}	60	50	40

still dissipating energy. A plot of the minimum spatial average mean square motion as a function of the associated optimal damper location is given in Fig. 4. A quick examination of this data indicates that the absolute minimum value occurs when $a/L = 0.7$ for a dimensionless damper value of $C = 60$.

Similar calculations were carried out for cases (b) and (c) and data similar to that presented in Fig. 4 are shown in Figs. 5 and 6 and it is also clear that in these cases there is an optimal location and damping parameter. These optimal parameters are shown in Table 2.

The purpose of considering the three cases was to see how strongly the spatial distribution of the forcing function influences the optimal values and it is seen from Table 2 that a good selection for a/L is about 0.7 and that for the dimensionless damping coefficient is $C = 50$.

4. Conclusion

In the work reported here it is shown that the response of a cantilever beam to temporally white noise with three spatial distributions by means of an attached viscous damper can be minimized in the sense of the spatial average mean square motion by proper selection of the damper value and the point of attachment. For each point of attachment there is a value of the damper parameter which will minimize the spatial average of the mean square response. Among all these values there is a location for which there exists a global minimum. For the variety of spatial distributions of forcing functions considered the best location is about 70% of the length from the fixed end and the value of the damper coefficient is about $b = 50\rho AL\omega_1$.

Acknowledgements

The author would like to thank the very perceptive reviewers for their careful review and insightful comments. In response to them the manuscript was improved considerably.

References

[1] J.T. Wissenberger, Effect of local modifications on vibration characteristics of linear systems, Journal of Applied Mechanics 35 (1968) 327–332.
[2] R.G. Jacquot, W. Soedel, Vibration of elastic surface systems carrying dynamic elements, Journal of the Acoustical Society of America 47 (1970) 1354–1358.
[3] R.J. Pomazal, V.W. Snyder, Local modifications of damped linear systems, American Institute of Aeronautics and Astronautics Journal 9 (1971) 2216–2221.
[4] E.H. Dowell, Free vibration of an arbitrary structure in terms of component modes, American Society of Mechanical Engineers, Journal of Applied Mechanics 39 (1972) 727–732.
[5] J. Hallquist, V.W. Snyder, Linear damped vibratory structures with arbitrary support conditions, Journal of Applied Mechanics 40 (1973) 312–313.
[6] L.J. Howell, Effects of constraint modification on random vibration of damped linear structures, American Institute of Aeronautics and Astronautics Journal 13 (1975) 685–687.
[7] L.A. Bergman, J.W. Nicholson, Stationary response of combined linear dynamic systems to stationary excitation, Journal of Sound and Vibration 103 (2) (1985) 225–236.
[8] A. Kareem, W.-J. Sun, Stochastic response of structures with fluid-containing appendages, Journal of Sound and Vibration 119 (3) (1987) 389–408.
[9] D. Yadav, S. Kamle, S. Talukdar, Transverse vibration of multimass loaded variable section beams under random excitation, Journal of Sound and Vibration 207 (2) (1996) 151–173.
[10] R.G. Jacquot, Random vibration of damped modified beam systems, Journal of Sound and Vibration 243 (3) (2000) 441–454.
[11] R.G. Jacquot, Suppression of random vibration in plates using vibration absorbers, Journal of Sound and Vibration 248 (4) (2001) 585–596.
[12] R.G. Jacquot, The spatial average mean square motion as an objective function for optimizing damping in damped modified systems, Journal of Sound and Vibration 259 (4) (2003) 955–965.
[13] B.M. Pacheco, Y. Fujino, A. Sulekh, Estimation curve for modal damping in stay cables with viscous damper, Journal of Structural Engineering 119 (6) (1993) 1961–1979.
[14] E.A. Johnson, B.F. Spencer, Y. Fujino, Semiactive damping of stay cables: a preliminary study, Proceedings of the 1999 17th International Modal Analysis Conference-IMAC Part 1, 1999, pp. 417–423.
[15] D. Young, R.P. Felgar, Tables of Characteristic Functions Representing Normal Modes of a Vibrating Beam, University of Texas Bureau of Engineering Research, Austin, TX, 1949.
[16] R.P. Felgar, Formulas for Integrals Containing Characteristic Functions of a Vibrating Beam, University of Texas Bureau of Engineering Research, Austin, TX, 1950.